Numerical Treatment of Energy and Potential Vorticity Conservation on Arbitrarily-Structured C-Grids

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Climate, Ocean and Sea-Ice Modeling Project
http://public.lanl.gov/ringler/ringler.html

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Cutting to the chase ....

<u>Analytic results for the nonlinear shallow-water equations:</u>

1. Stationary geostrophic mode is recovered.

2. Total energy is conserved to within time truncation.
   a. Coriolis force is energetically-neutral
   b. Transport of KE is conservative
   c. KE/PE exchange is equal and opposite.

3. Potential vorticity is conserved to round-off. PV is compatible with an underlying thickness evolution equation.

4. It appears* that potential enstrophy can be dissipated.

Results hold for a wide class of meshes: Lat/Lon, Stretched Lat/Lon, Voronoi Tessellations, Delaunay Triangulation and Conformally-mapped cubed sphere meshes.

Equation Set ....

PDE:

$$\frac{\partial h}{\partial t} + \nabla \cdot (h\mathbf{u}) = 0$$

$$\frac{\partial \mathbf{u}}{\partial t} + q(h\mathbf{u}^{\perp}) = -g\nabla\left(h + h_s\right) - \nabla K$$

definition:

$$\eta = \nabla \times \mathbf{u} + f$$

$$\mathbf{u}^{\perp} = \mathbf{k} \times \mathbf{u}$$

$$q = \frac{\eta}{h}$$

Relationship between nonlinear Coriolis force and potential vorticity flux

nonlinear Coriolis force

$$\mathbf{k} \cdot \nabla \times \left[\frac{\partial \mathbf{u}}{\partial t} + q(h\mathbf{u}^{\perp}) = -g\nabla\left(h + h_s\right) - \nabla K \right]$$

$$\frac{\partial \eta}{\partial t} + \mathbf{k} \cdot \nabla \times \left[\eta \mathbf{u}^{\perp}\right] = 0$$

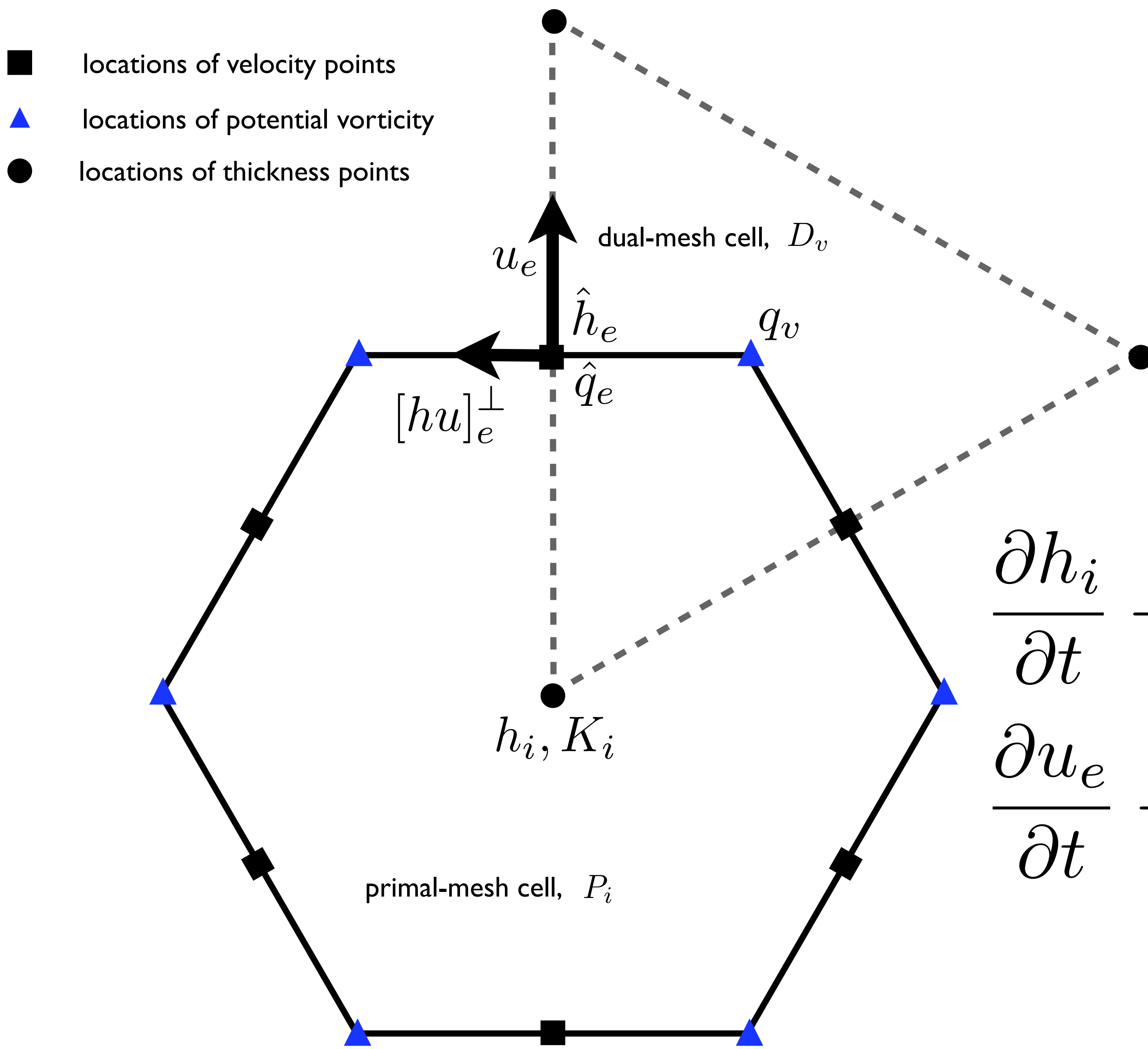
$$\frac{\partial \eta}{\partial t} + \nabla \cdot \left[\eta \mathbf{u}\right] = 0$$

potential vorticity flux

$$\frac{\partial (hq)}{\partial t} + \nabla \cdot \left[hq\mathbf{u}\right] = 0$$

The nonlinear Coriolis force IS the PV flux in the direction perpendicular to the velocity.

Defining the discrete system

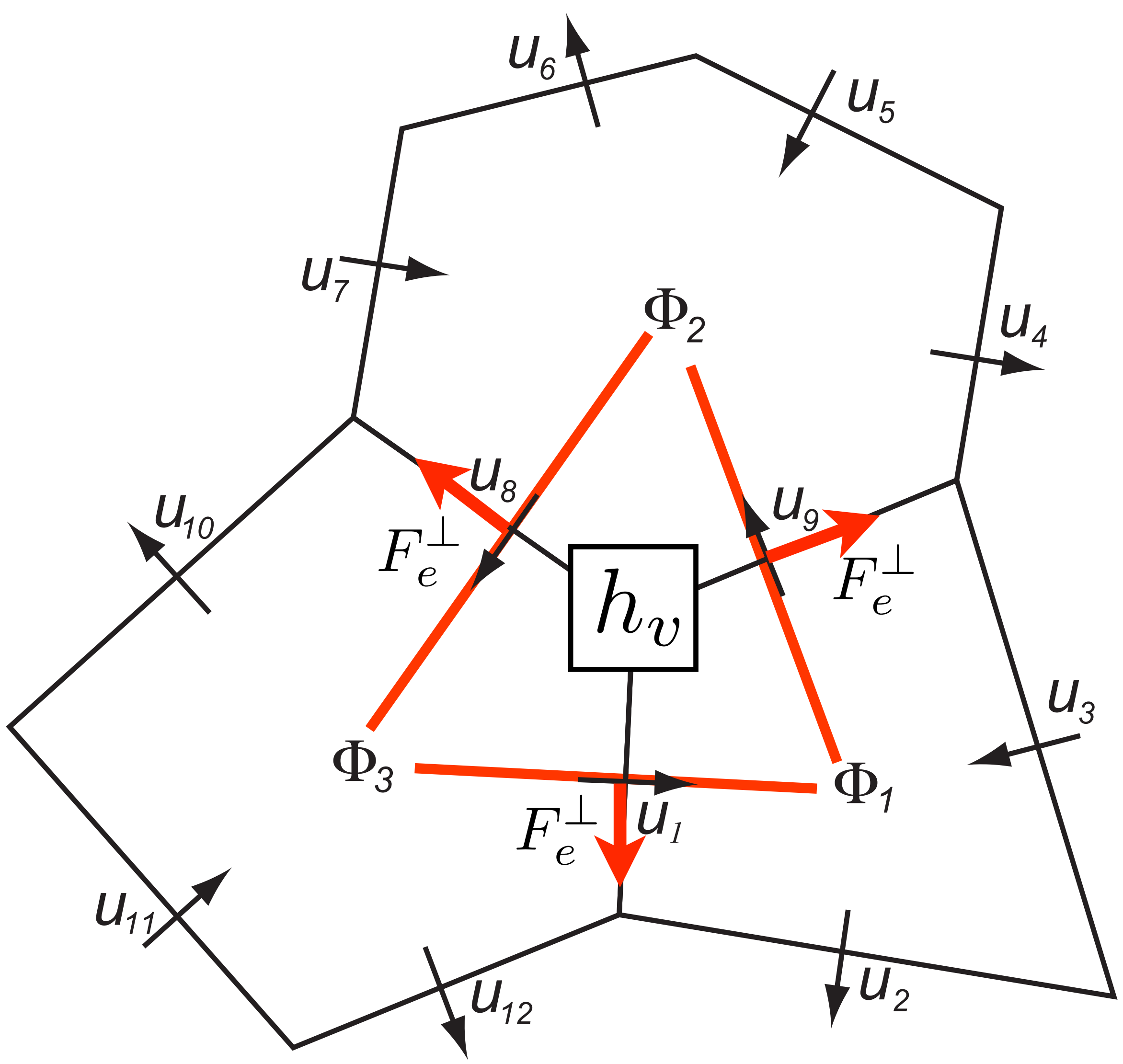


$$\frac{\partial h_i}{\partial t} + \left[\nabla \cdot \left(\widehat{h}_e u_e\right)\right]_i = 0$$

$$\frac{\partial u_e}{\partial t} + \widehat{q}_e \, [hu]_e^\perp = \left[\nabla \left(g h_i + K_i\right)\right]_e$$

Deriving an auxiliary vertex thickness equation

(Because PV lives on vertices and PV means nothing without a thickness equation)



$$\frac{\partial h_i}{\partial t} + \left[\nabla \cdot \left(\widehat{h}_e u_e\right)\right]_i = 0$$

$$\frac{\partial h_i}{\partial t} + [\nabla \cdot F_e]_i = 0$$

$$d_e\, F_e^\perp = \sum_j w_e^j\, l_j\, F_j$$

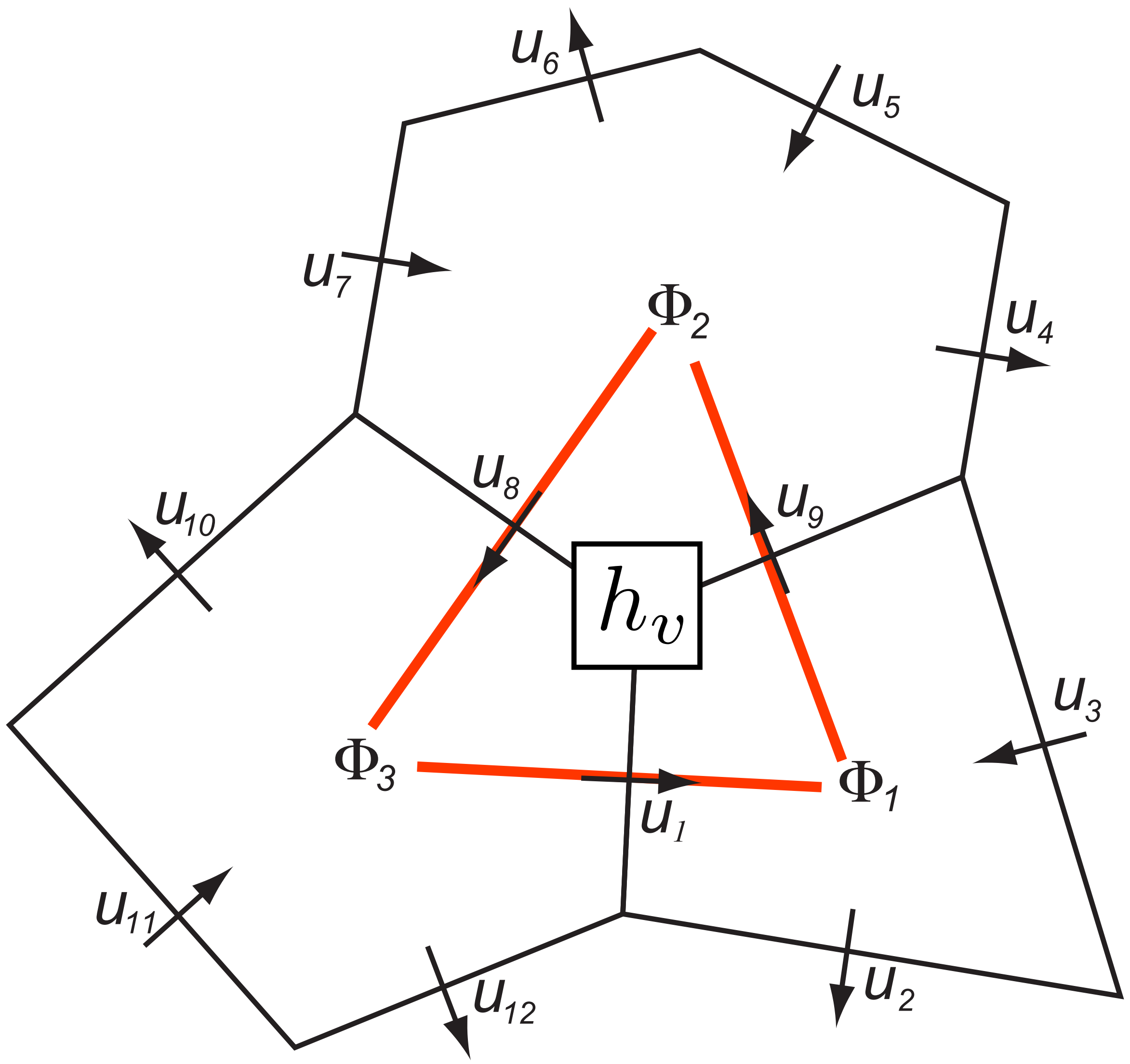
$$\left[\nabla \cdot F_e^\perp\right]_v \equiv \delta_v^{F^\perp} = \sum_{i \in G(v)} b_v^i \delta_i^F$$

$$\sum_{i \in G(v)} b_v^i = 1 \quad and \quad b_v^i \geq 0 \; \forall \; i, v$$

$$\delta_v^{F^\perp} = I[\delta_i^F]$$

Deriving an auxiliary vertex thickness equation

(Because PV lives on vertices and PV means nothing without a thickness equation)



$$\delta_v^{F^\perp} = I[\delta_i^F]$$

$$\delta_i^F \equiv \left[\nabla \cdot \left(\widehat{h}_e u_e\right)\right]_i$$

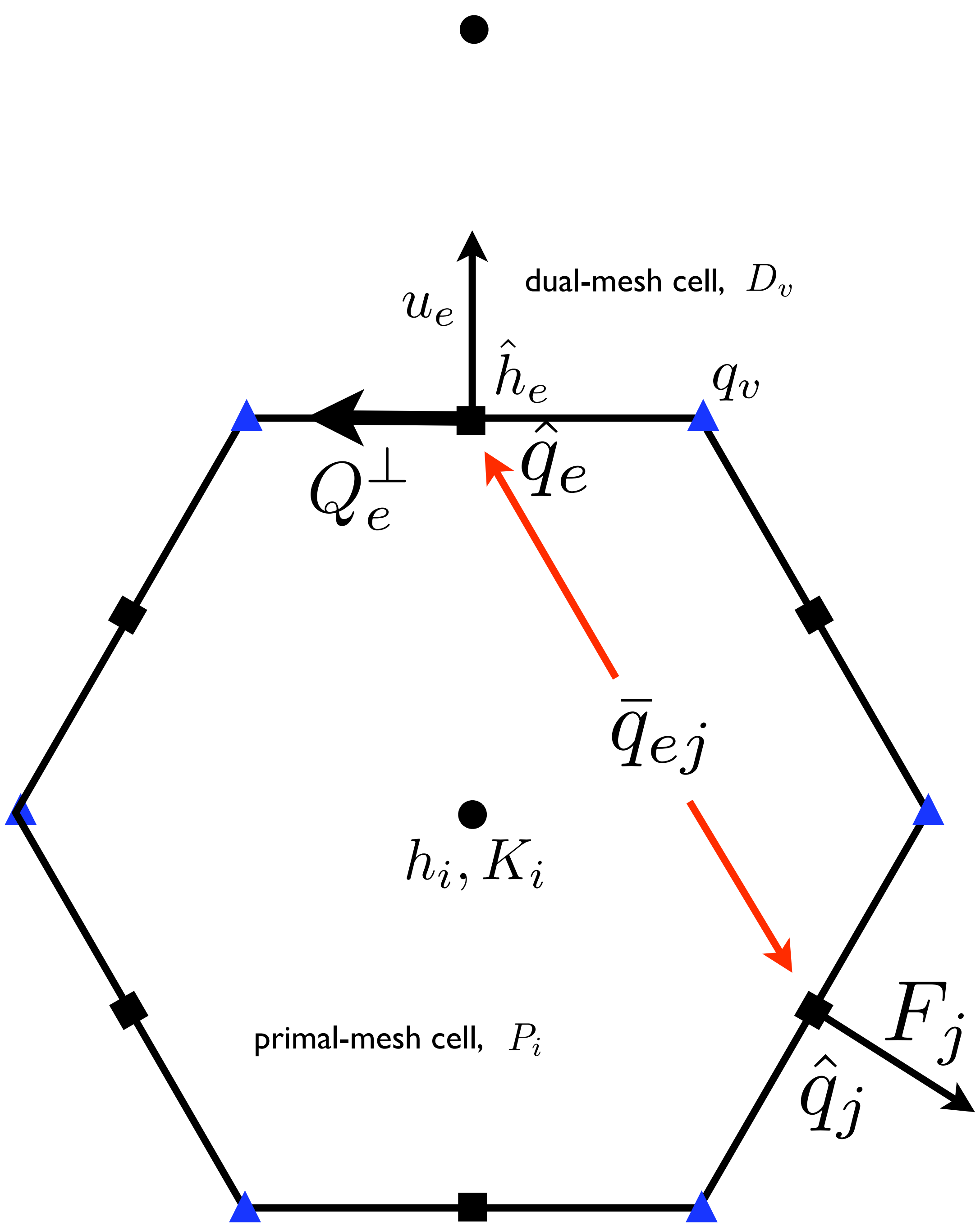
$$\frac{\partial h_i}{\partial t} = -\delta_i^F$$

$$\frac{\partial h_v}{\partial t} = -I\left[\delta_i^F\right] = I\left[\frac{\partial h_i}{\partial t}\right]$$

If $h_v(t=0) = I(h_i(t=0))$, then h_v is bounded by neighboring h_i for all time.

Reconstructing the nonlinear Coriolis force

(recall that the nonlinear Coriolis force is the the PV-flux perpendicular to the velocity)



$$\frac{\partial u_e}{\partial t} + \widehat{q}_e \left[hu\right]_e^\perp = \left[\nabla\left(gh_i + K_i\right)\right]_e$$

$$\frac{\partial u_e}{\partial t} + Q_e^\perp = \left[\nabla\left(gh_i + K_i\right)\right]_e$$

$$d_e\, Q_e^\perp = \sum_j w_e^j\, l_j\, F_j\, \bar{q}_{ej}$$

$F_j = \hat{h}_j u_j$ thickness flux

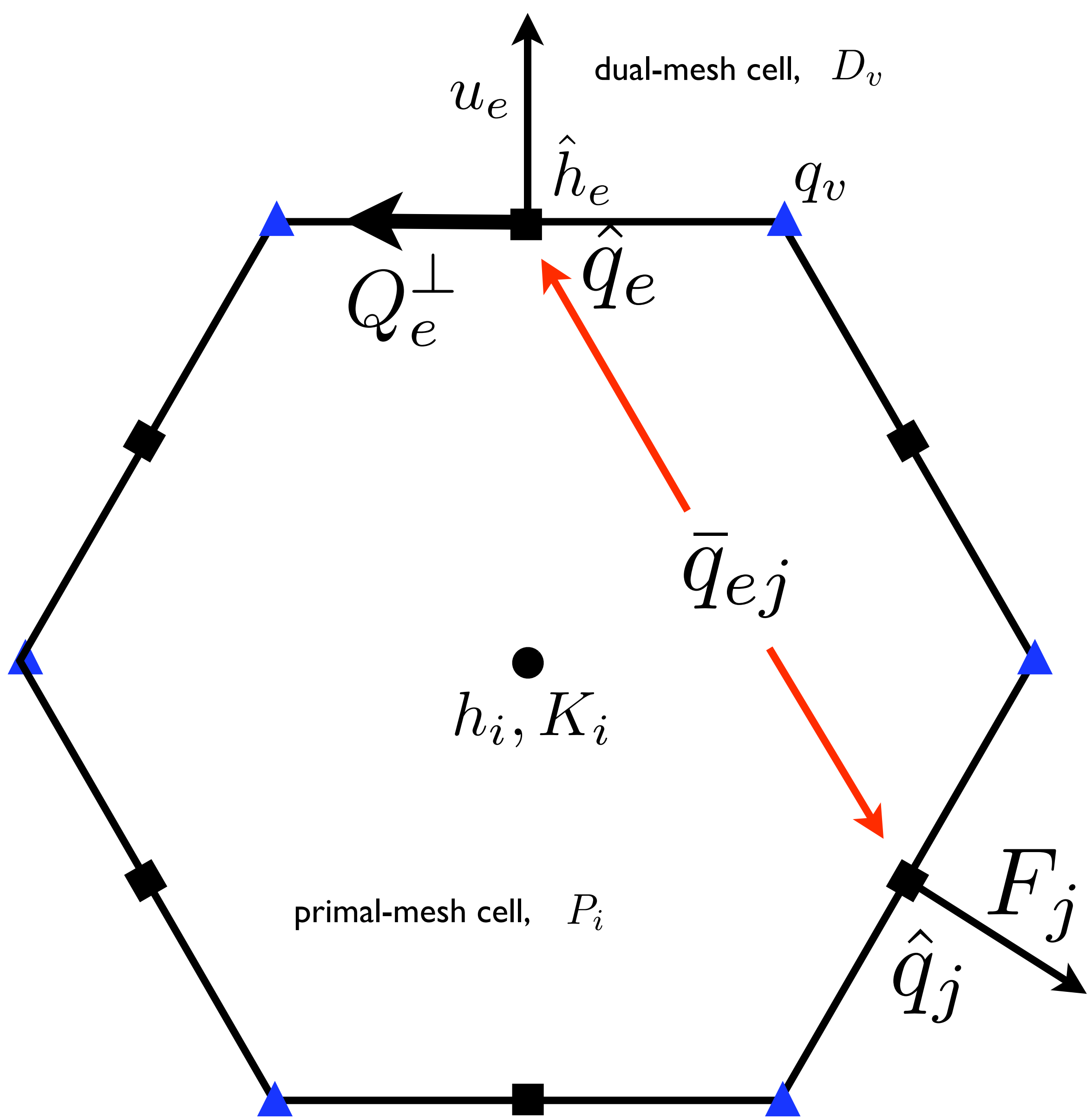
$w_e^j = -w_j^e$ weights are equal and opposite

$\bar{q}_{ej} = \bar{q}_{je}$ PV is symmetric

The nonlinear Coriolis force will be energetically neutral for any $\bar{q}_{ej}$.
This is an extension to what Sadourny (1975) showed for regular meshes.

Reconstructing the nonlinear Coriolis force

(recall that the nonlinear Coriolis force is the the PV-flux perpendicular to the velocity)



$$\frac{\partial u_e}{\partial t} + \widehat{q}_e \, [hu]_e^\perp = [\nabla (gh_i + K_i)]_e$$

$$\frac{\partial u_e}{\partial t} + Q_e^\perp = [\nabla (gh_i + K_i)]_e$$

The curl of the above eq, lead to the below eq.

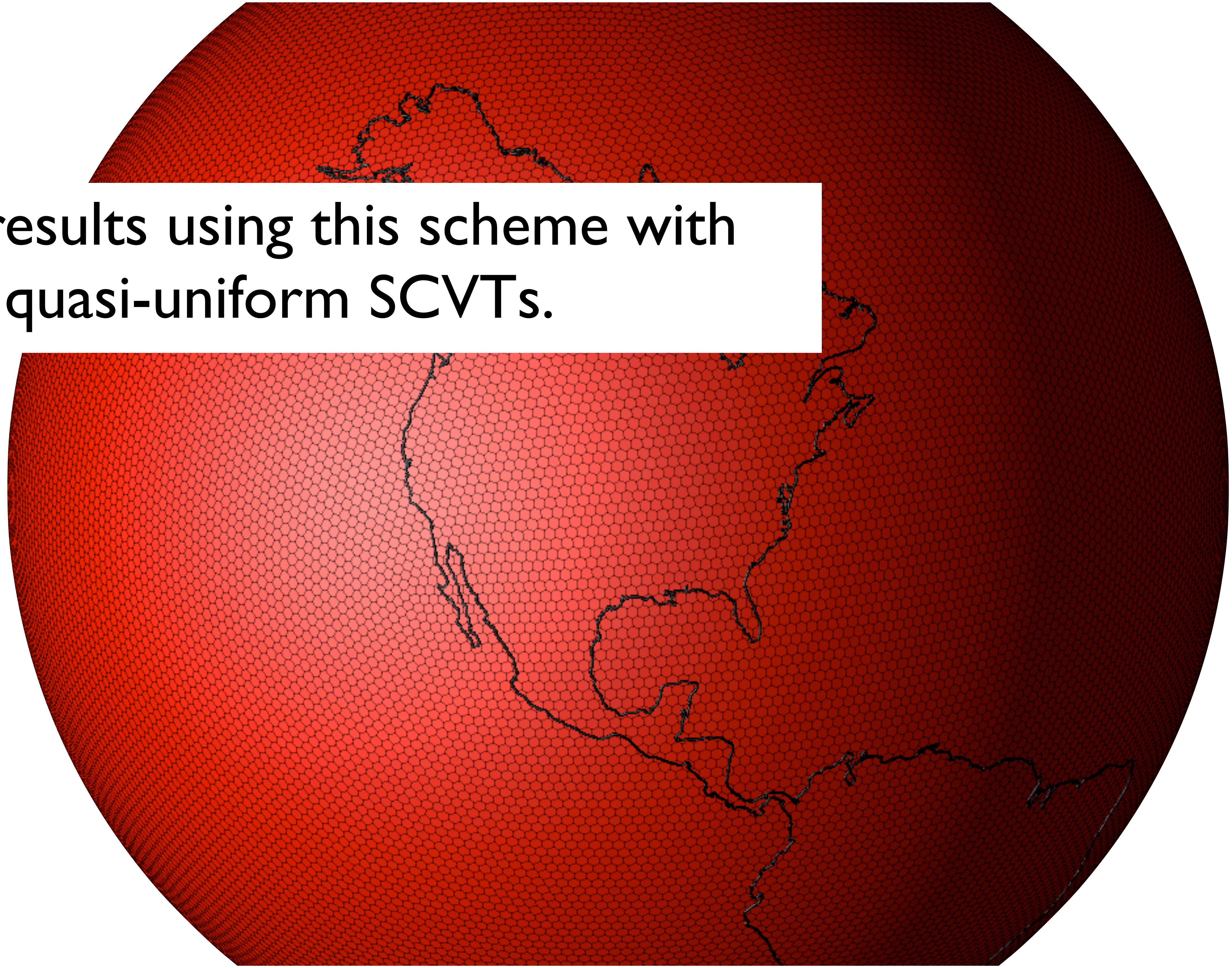
$$\frac{\partial}{\partial t}(h_v q_v) + \frac{1}{A_v} \sum_{e \in G(v)} Q_e^\perp \, dc_e = 0$$

For a uniform PV field, the above eq reduces (identically) to the vertex thickness eq.

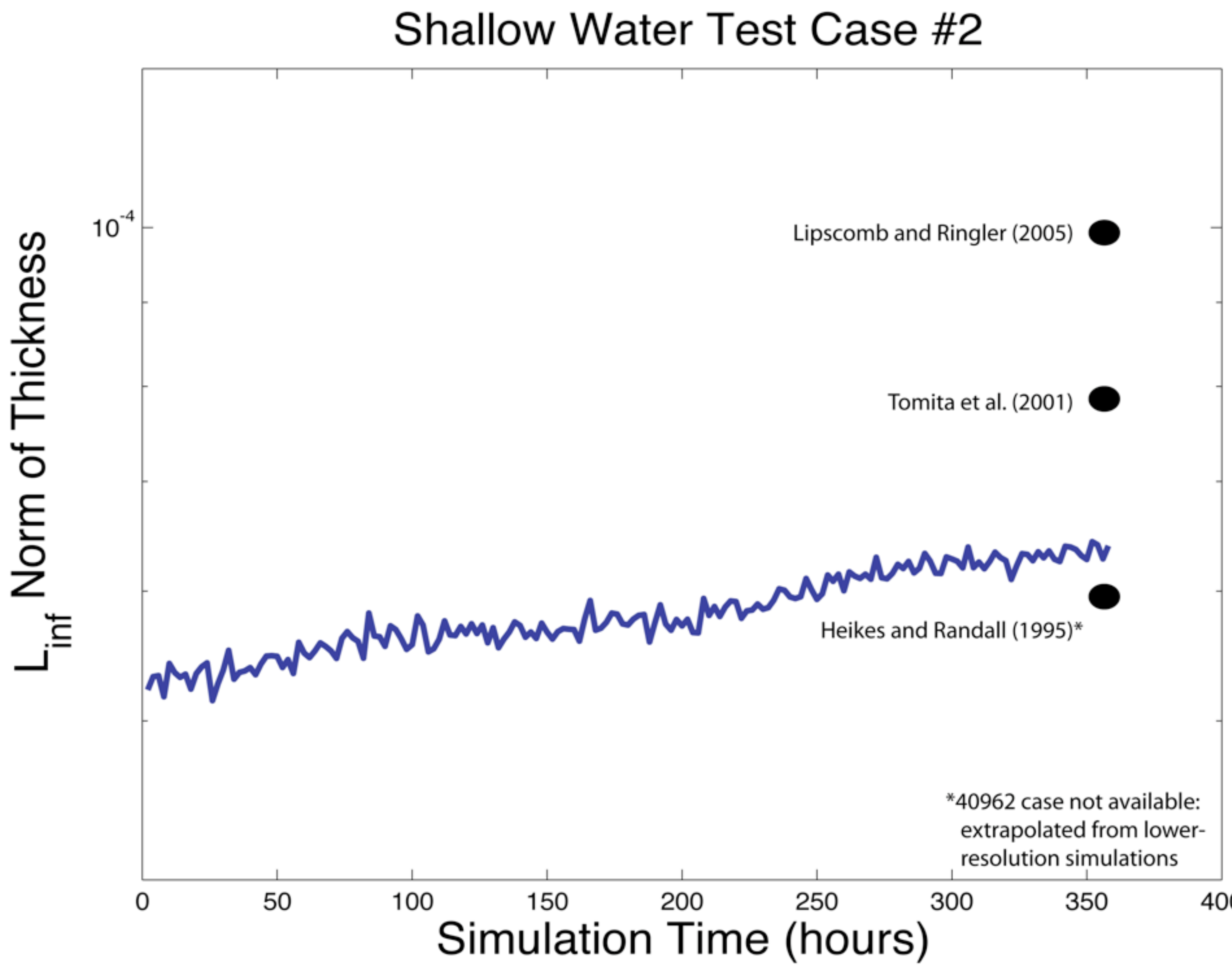
$$\frac{\partial}{\partial t}(h_v) + \frac{1}{A_v} \sum_{e \in G(v)} F_e^\perp \, dc_e = 0$$

The evolution of the discrete velocity field is compatible with the evolution of a valid, discrete PV equation. The compatibility holds to round-off.

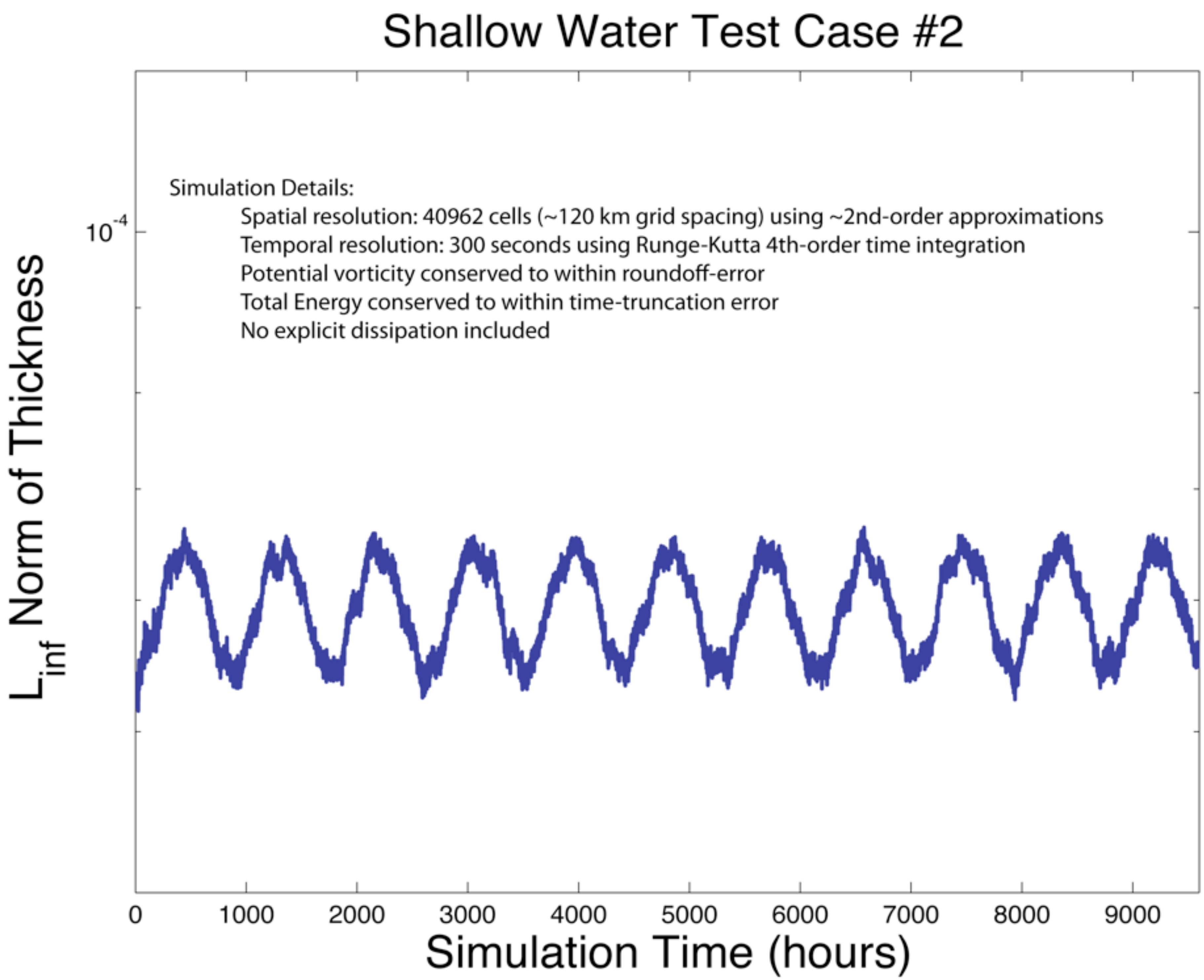
Some results using this scheme with quasi-uniform SCVTs.



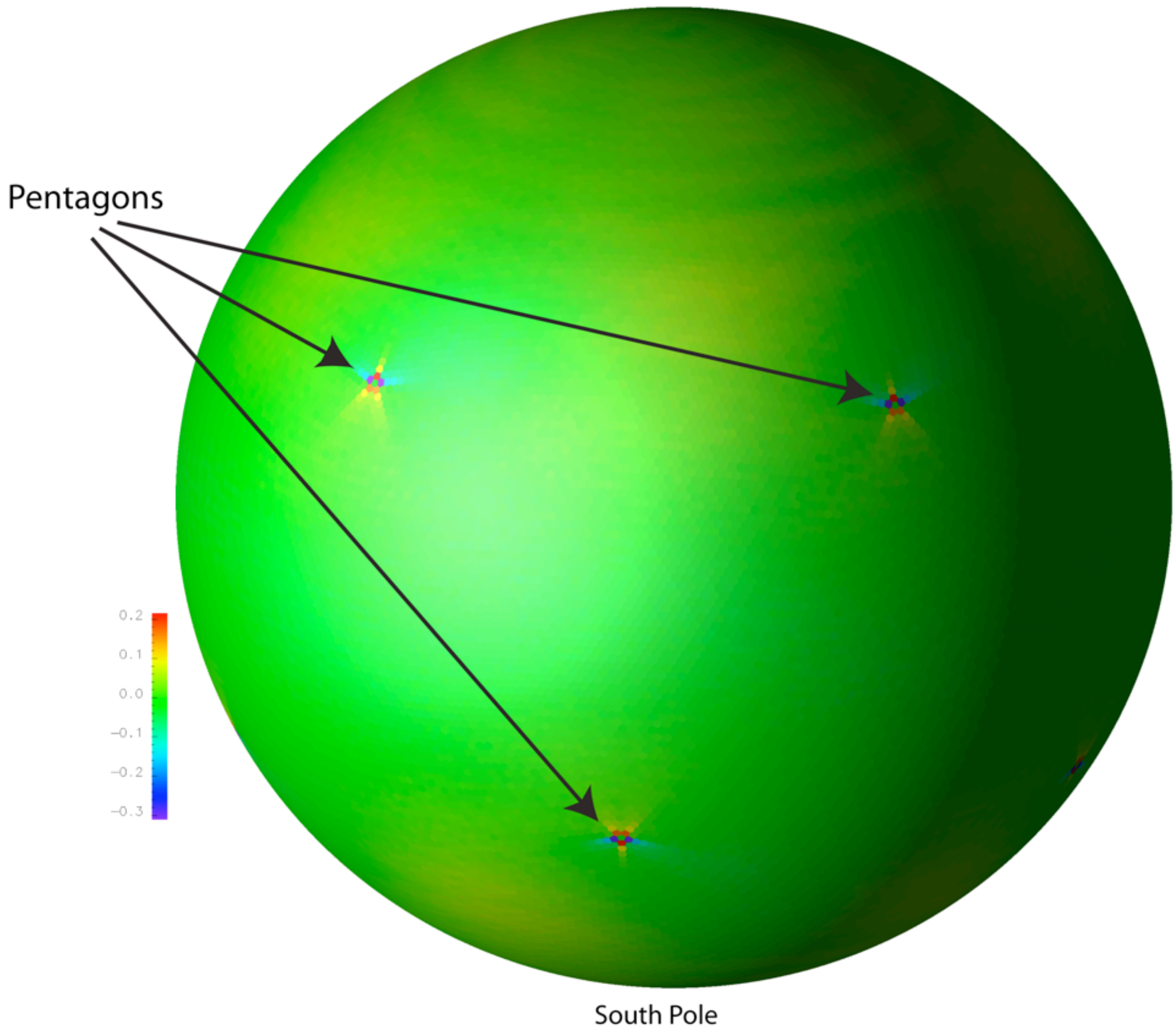
Quasi-Uniform 40962 mesh, ~120 km resolution



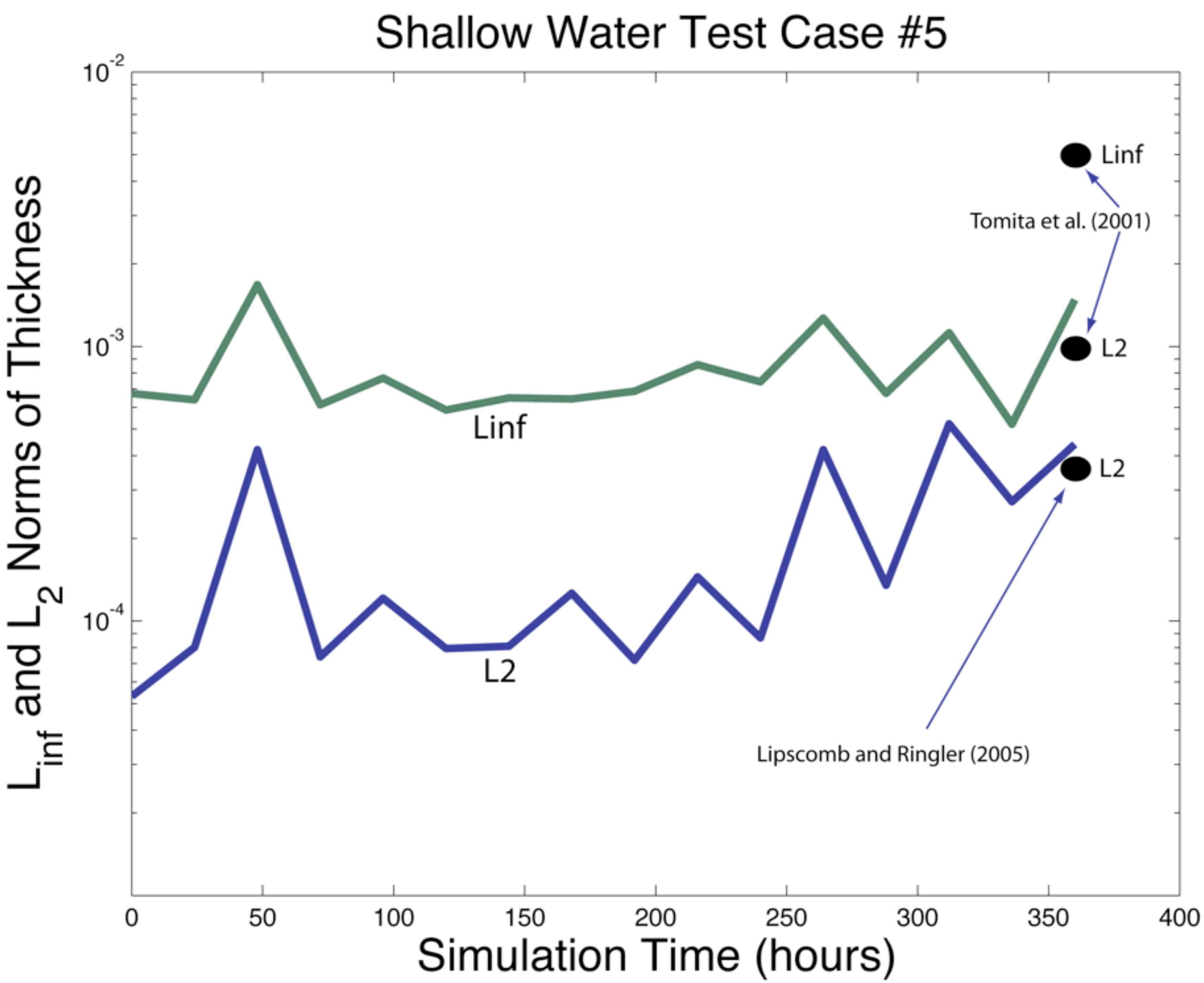
Quasi-Uniform 40962 mesh, ~120 km resolution



Error in thickness field (m) measured at day 400



Moving to SWTC#5 ....



Results of scheme with quasi-uniform meshes:
New scheme is competitive with existing models.

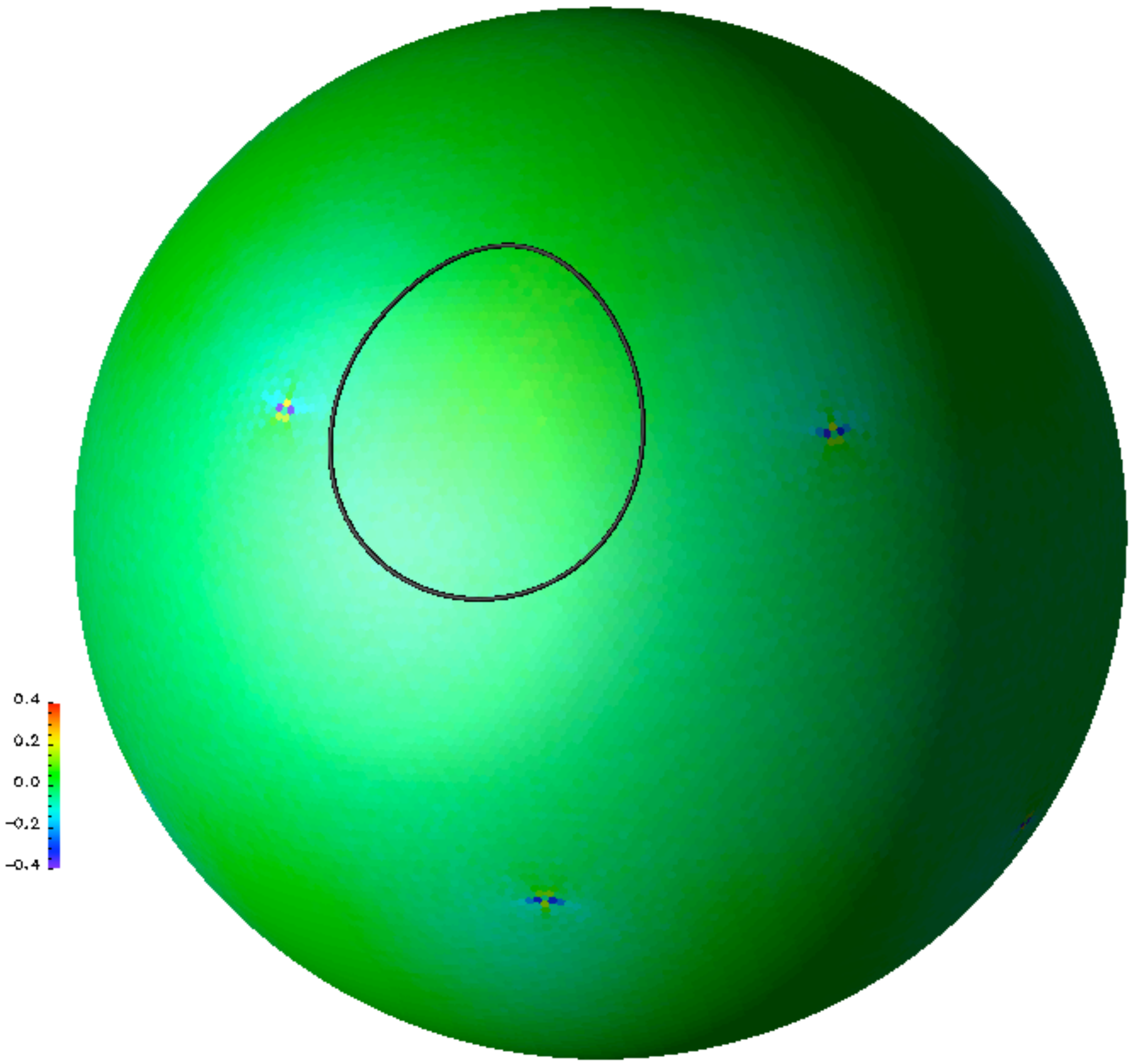
Some results from the nested, variable-resolution SCVT

Nested, Conformal, Variable-Resolution Meshes

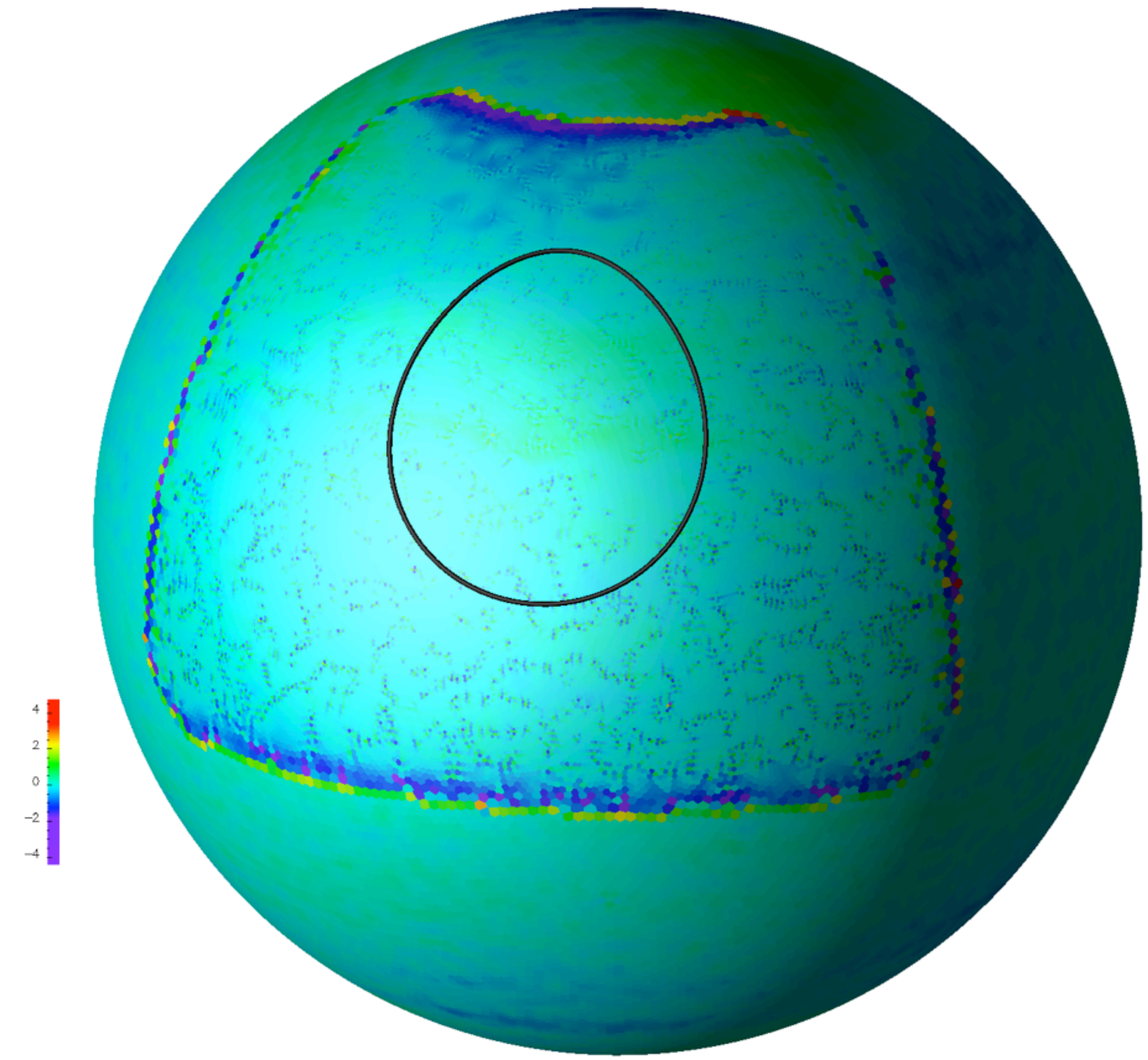
Simulation details:
RK4 time integration, centered-in-space numerics, no dissipation.

SWTC#2 at Day 50.

uniform 40962

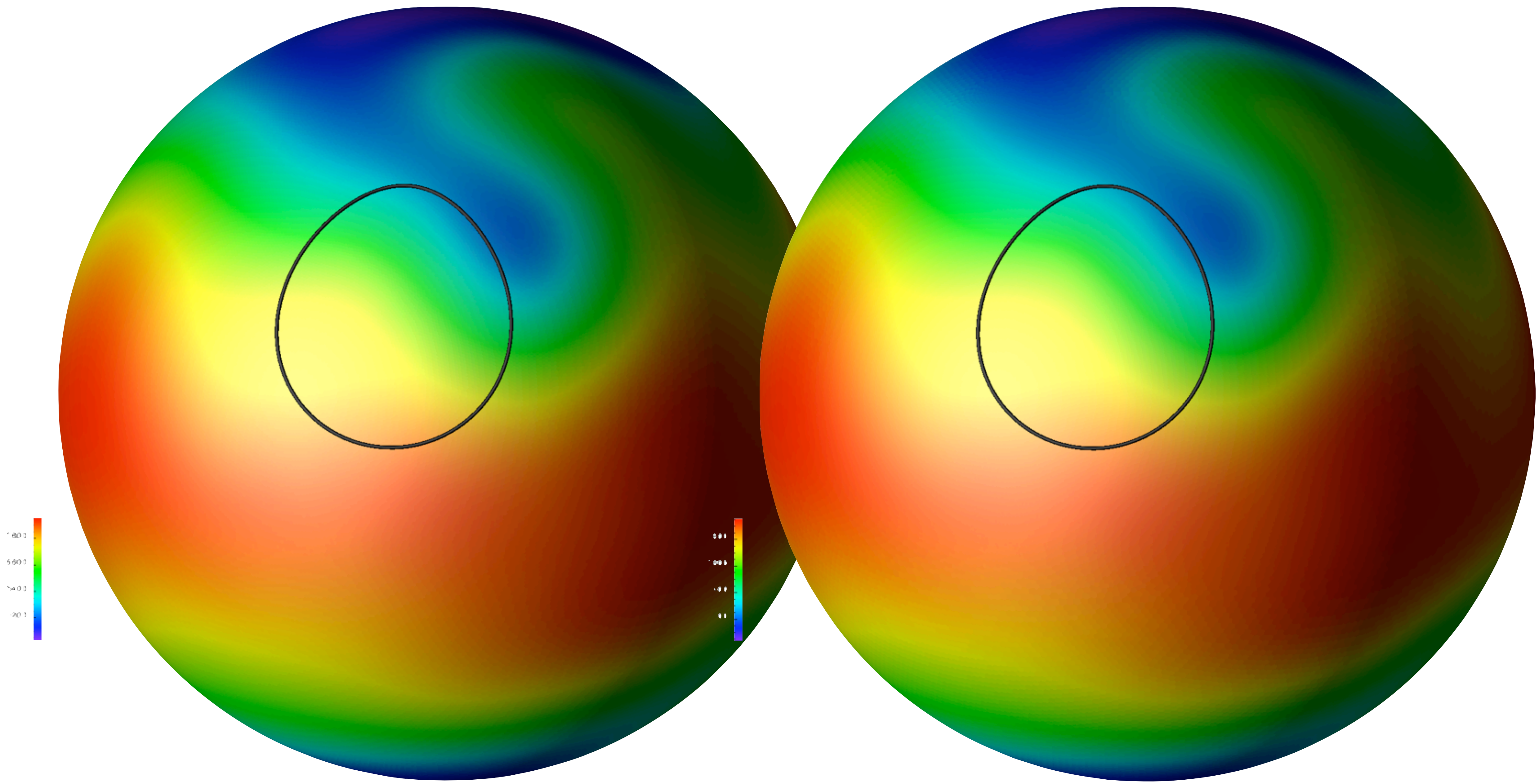


nested 82547



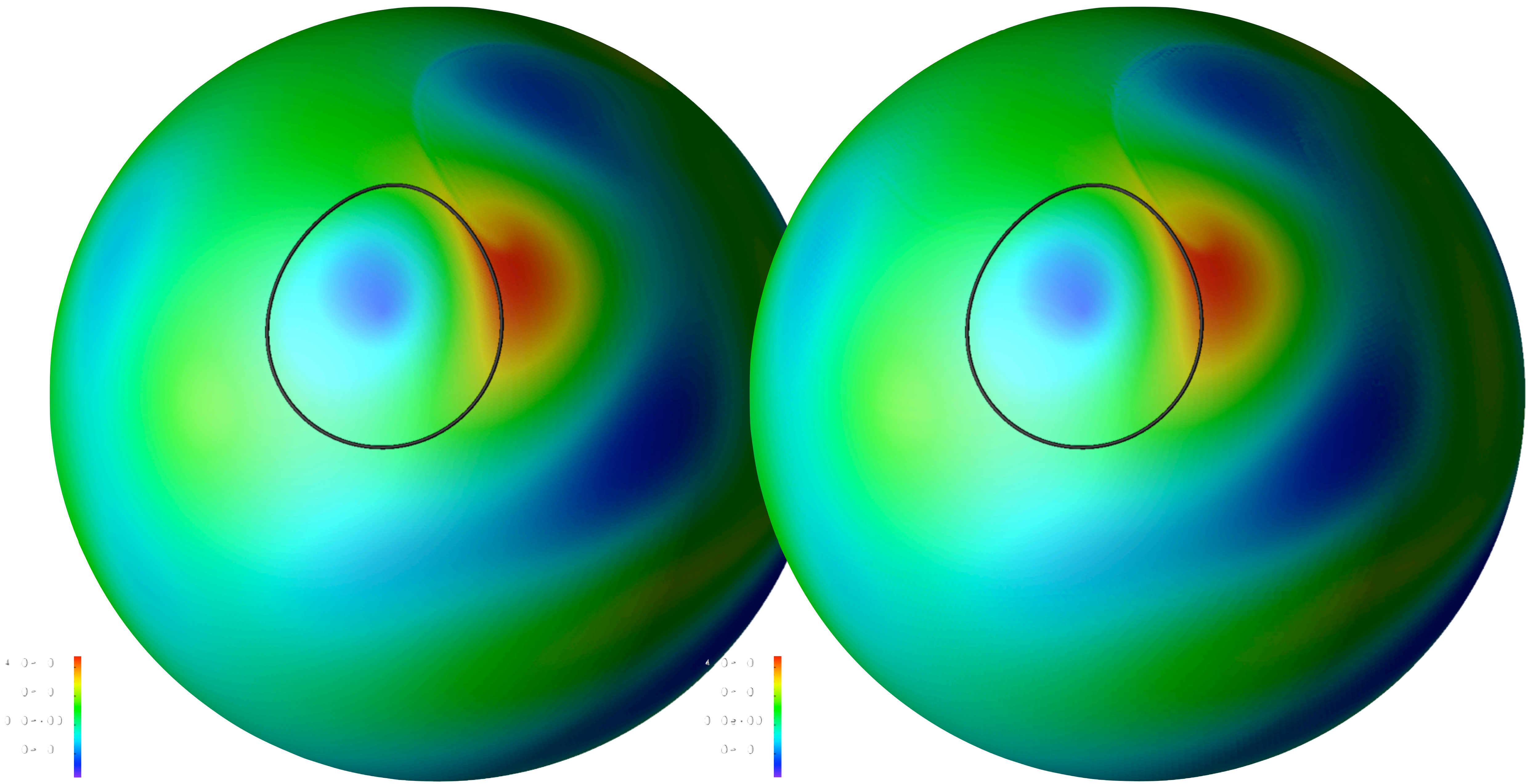
Maximum error is approximately a factor of 10 larger on the nested grid. Since this is the first nested mesh that we have constructed, I am guessing that we can bring this error down significantly.

SWTC#5: Day 15, thickness field (m)



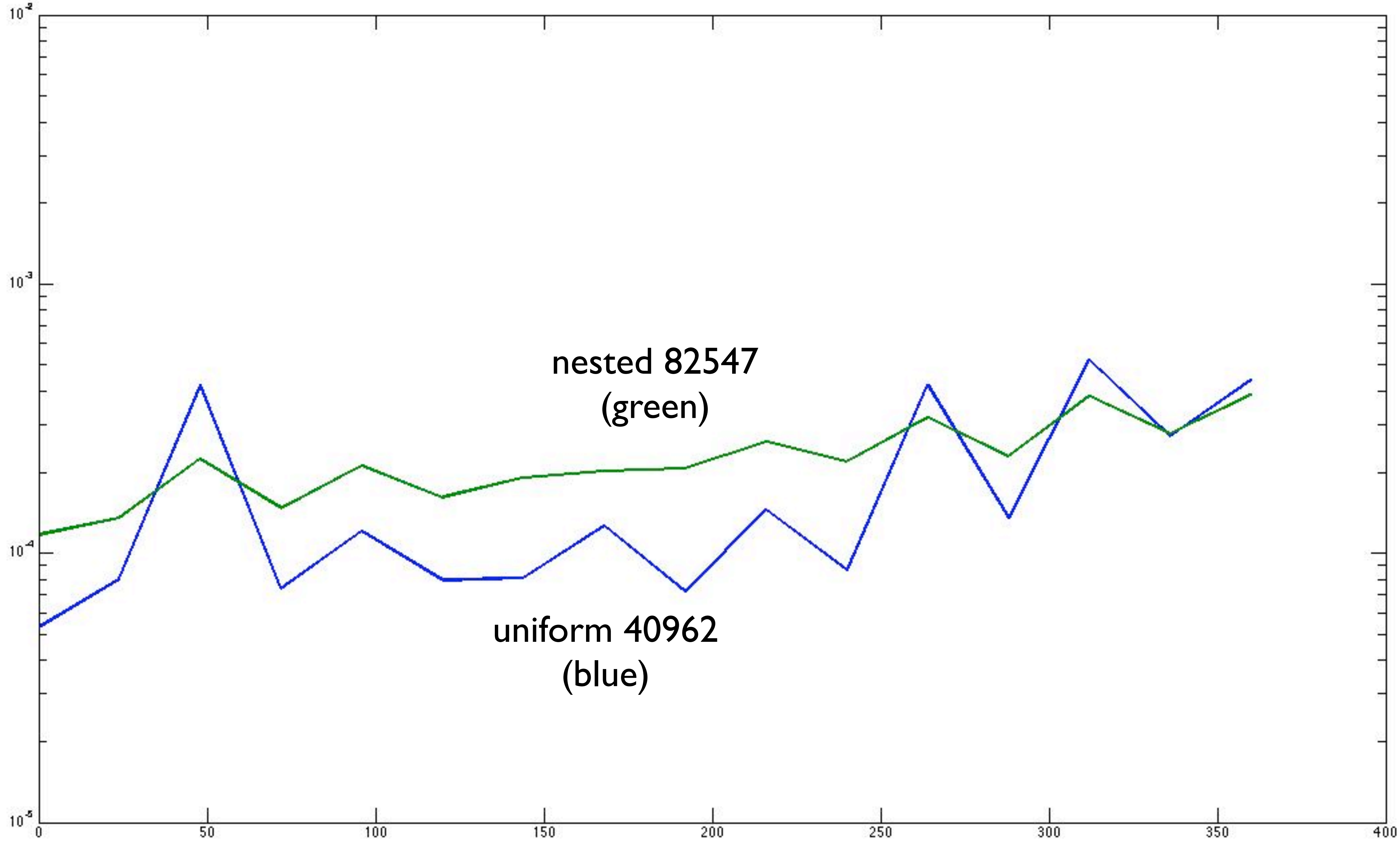
Indistinguishable.

SWTC#5: Day 15, relative vorticity (1/s)



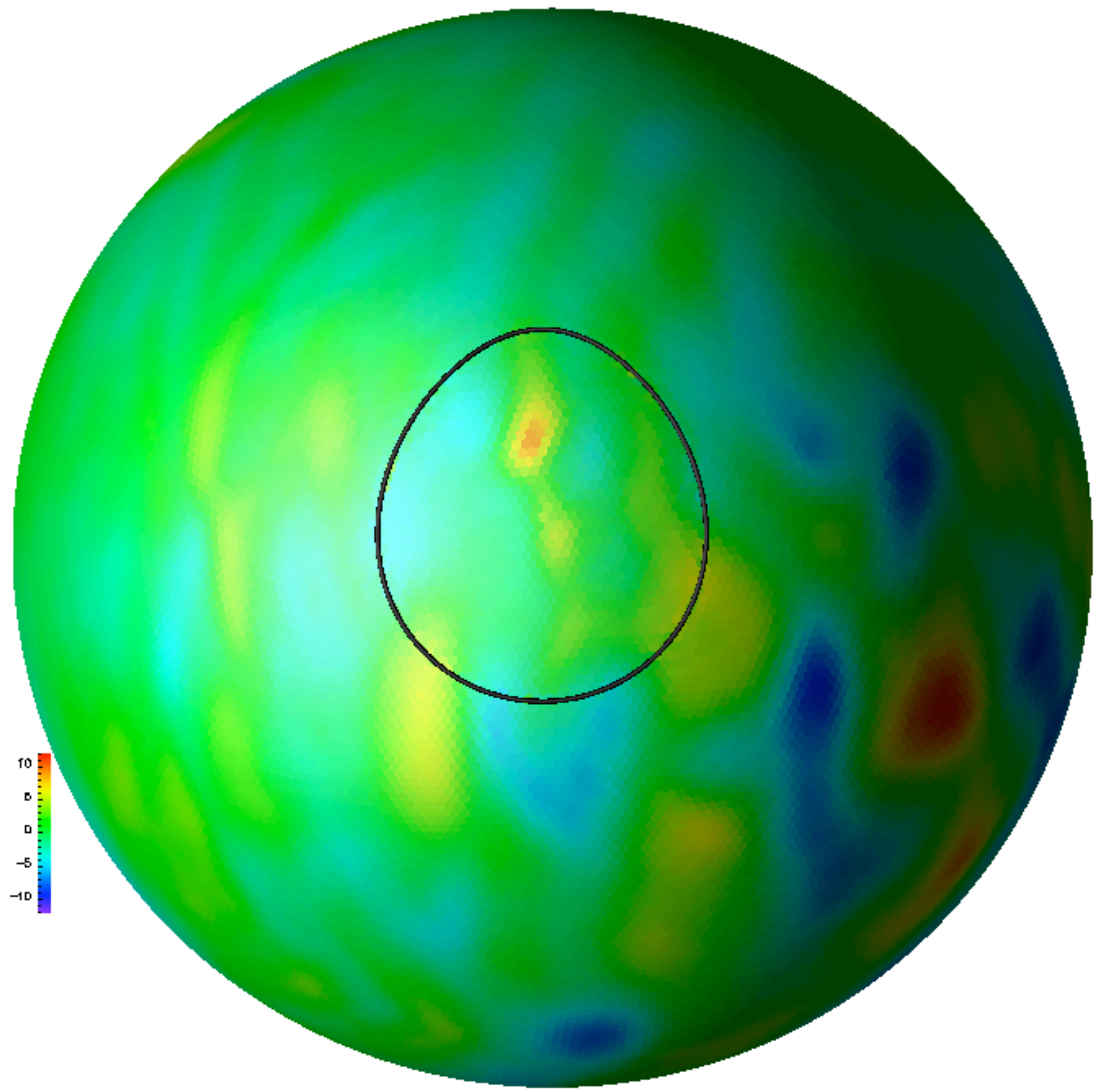
Essentially Indistinguishable.

SWTC#5: L2 error norm of thickness.



Results: Error Norms of thickness at day 15

uniform 40962



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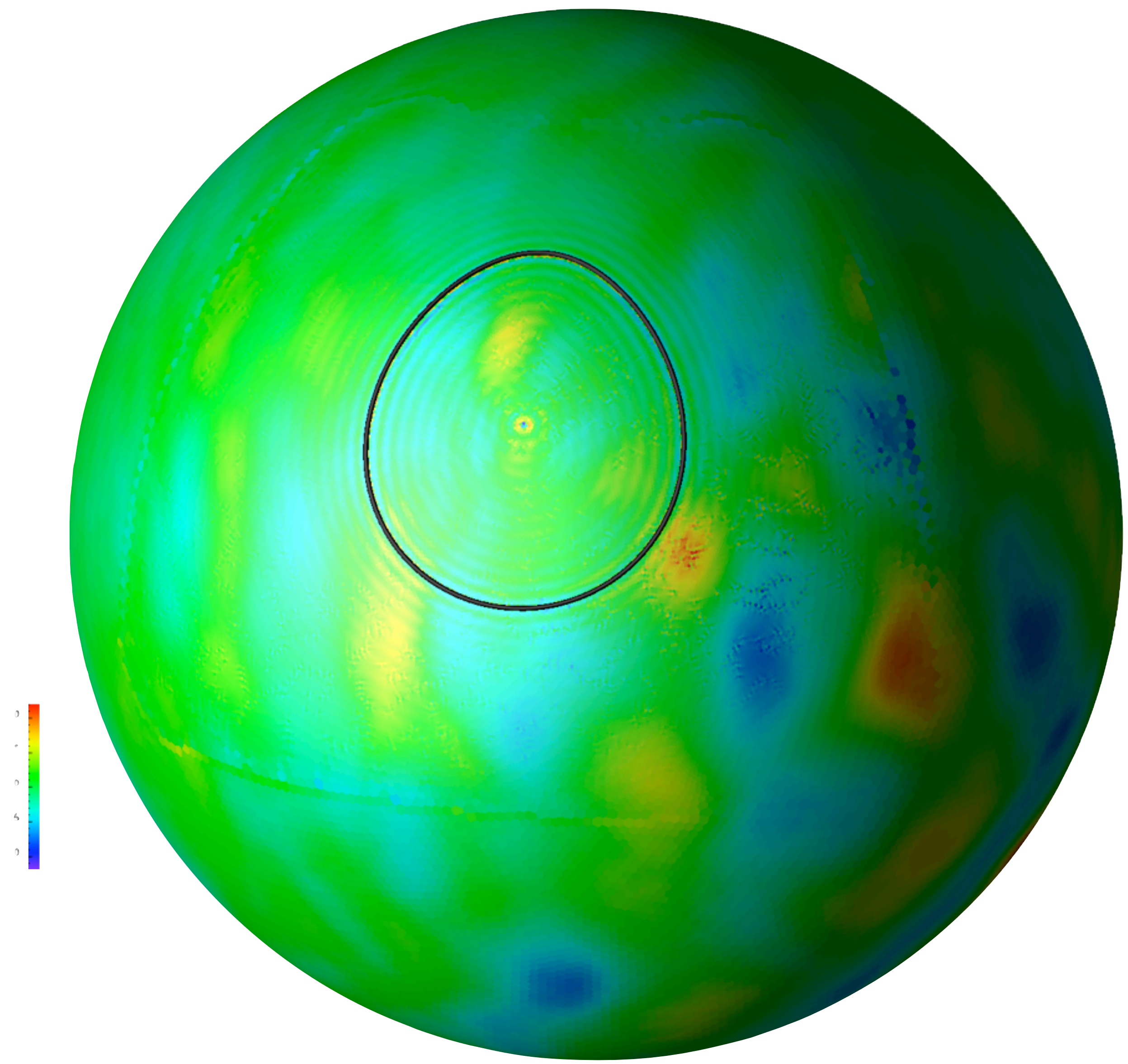
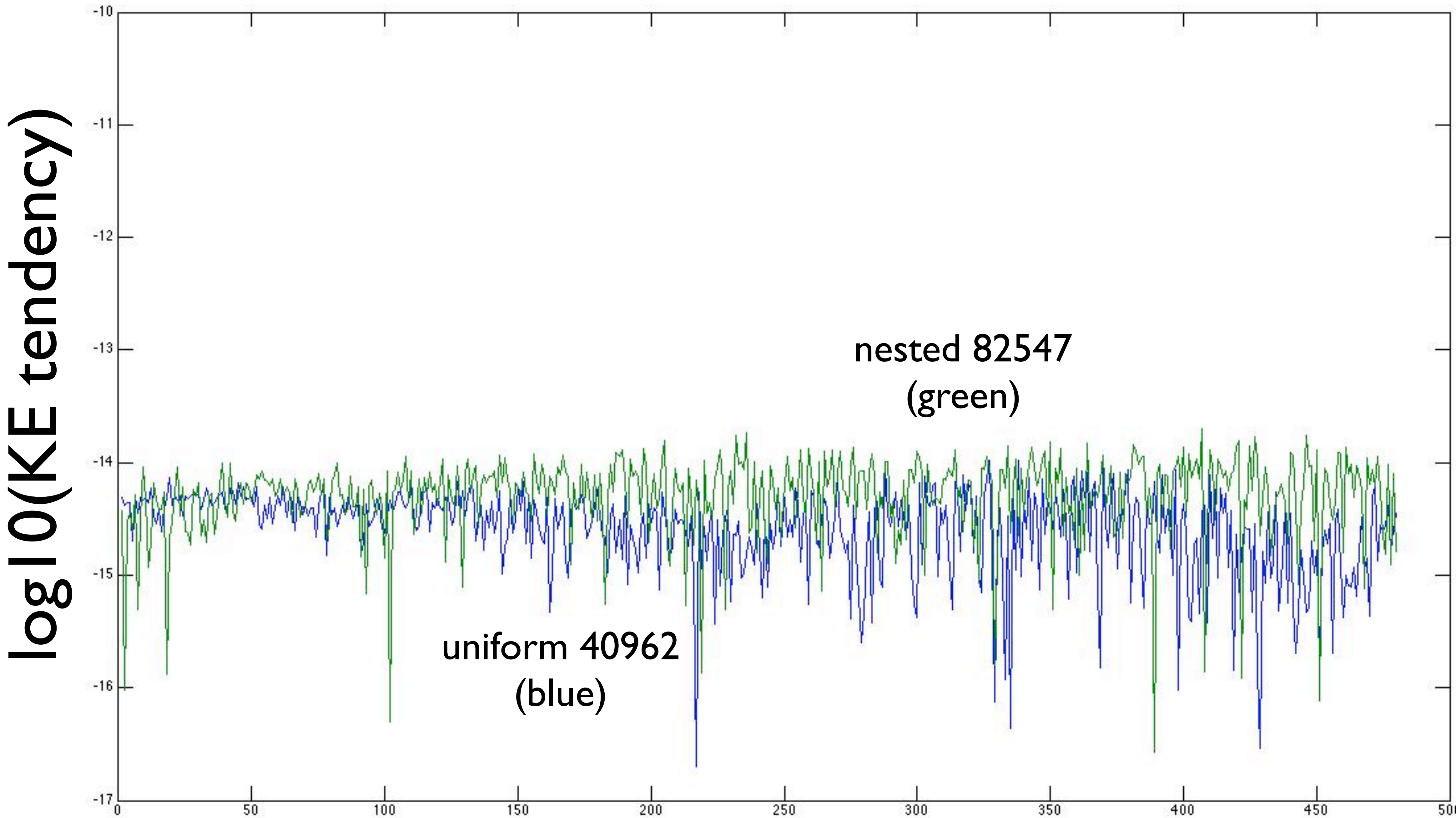
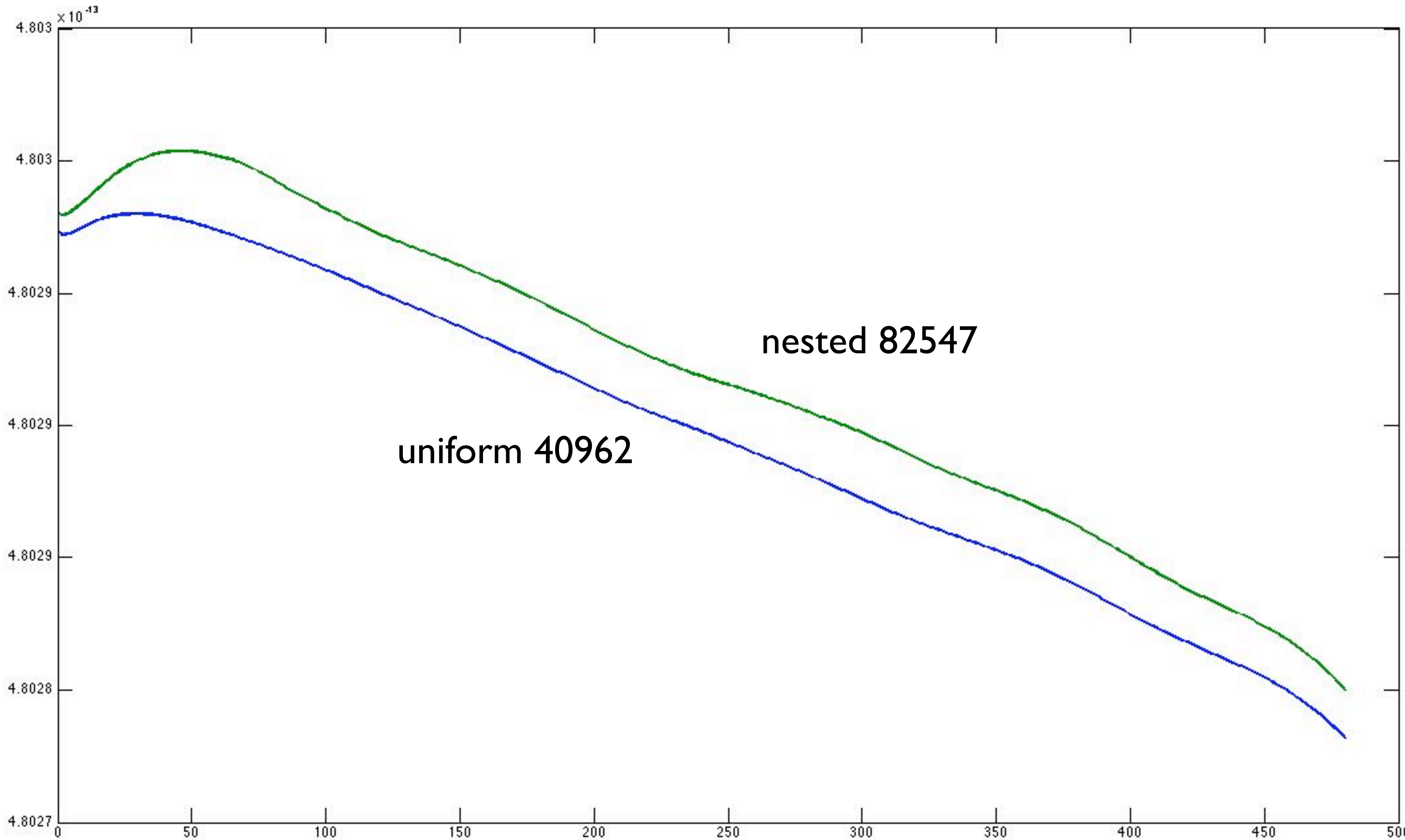


SWTC#5: Kinetic energy tendency due to nonlinear Coriolis term.



SWTC#5: Globally-averaged potential enstrophy evolution.



Results of scheme with variable resolution nested mesh: Preliminary results are very promising.

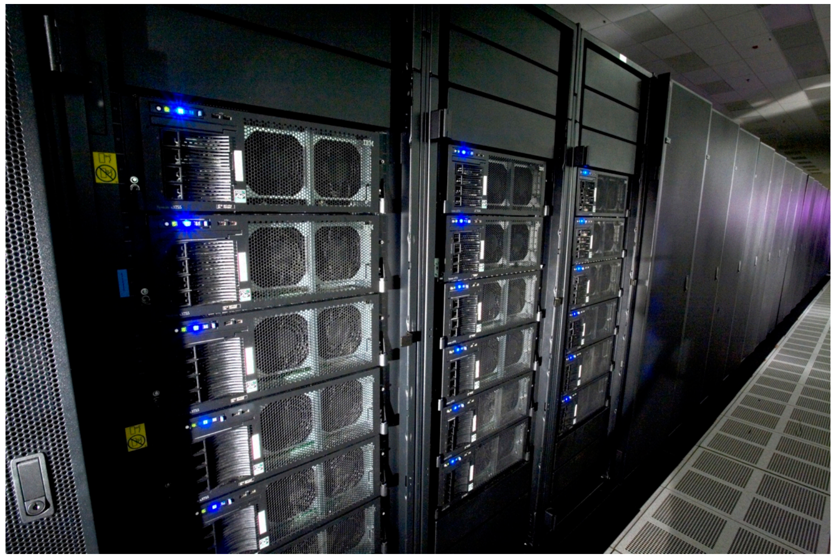
We think that this approach is analytically sound .... but can we get the throughput?

This is joint LANL / NCAR MMM development project.

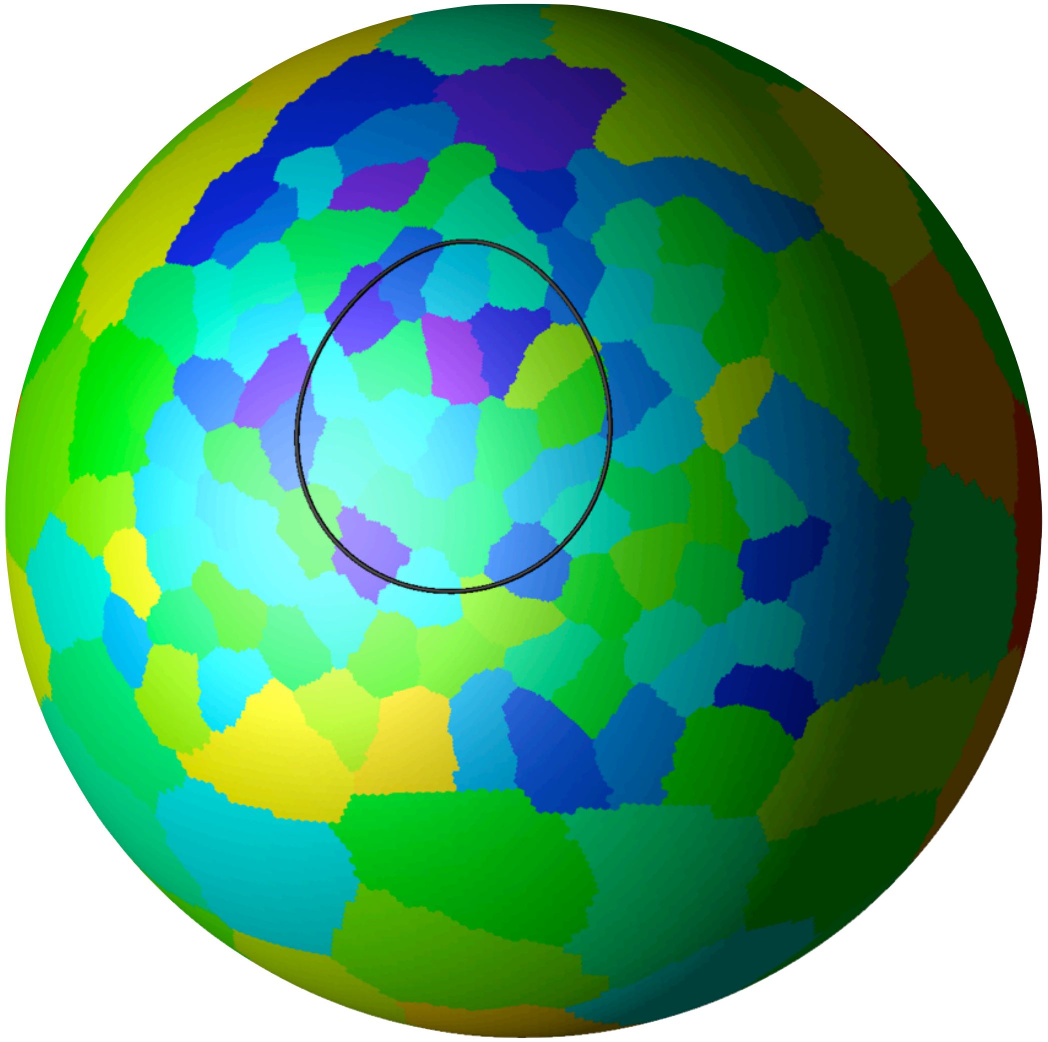
Approach:

The method requires an unstructured grid approach.
Targeting hybrid CPU/GPU machines first.
Start with a stacked shallow-water model.
Vectorize over the vertical layer index.
Group operators together and push to GPU.

Assuming that we can get the required CPU efficiency and scaling, LANL will likely pursue the construction of an ocean dynamical core and NCAR MMM will consider the construction of an atmosphere dynamical core.



LANL Cerrillos



Block decomposition:
Distribute blocks across processors
Maximize area to circumference ratio
Use for load balancing.

Summary

We have developed a numerical scheme suitable for climate simulation that is applicable to a wide class of meshes using C-grid staggering.

The results on quasi-uniform SCVTs are competitive with other FV schemes available.

The results on non-uniform SCVTs are promising. Long, stable and acceptably accurate simulations of the SW system without dissipation are possible.